

Mathematics Recap -

Linear Algebra

Why Linear Algebra?

Graphics → Transformation/Change of points and vectors.

"Matrices"

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3}$$

Matrix Arithmetic →

① Matrix Multiplication

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{im} \\ \vdots & & \vdots \\ a_{r1} & \dots & a_{rm} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1c} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \dots & b_{mj} & \dots & b_{mc} \end{bmatrix} = \begin{bmatrix} p_{11} & \dots & p_{1j} & \dots & p_{1c} \\ \vdots & & \vdots & & \vdots \\ p_{i1} & \dots & p_{ij} & \dots & p_{ic} \\ \vdots & & \vdots & & \vdots \\ p_{r1} & \dots & p_{rj} & \dots & p_{rc} \end{bmatrix}$$

Element $p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{im}b_{mj}$

Not commutative $\Rightarrow AB \neq BA$
 $AB=AC \quad B \neq C$ } Not necessarily

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

② Transpose A^T

$$a_{ij} = a_{ji}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$(AB)^T = B^T A^T$$

③ Determinants

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

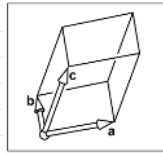
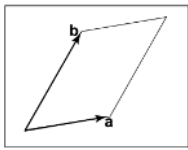
$$\det(A) = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det(A) = aei - afh - bdi + bfg + cdh - ceg$$

\uparrow $|A|$

Typically thought as
solving the system of equations.

In computer graphics,
as multiplication of vectors



$|ab| = \text{area of parallelogram}$

$|abc| = \text{volume of parallelepiped}$

④ Inverse matrix A^{-1}

$$\boxed{AA^{-1} = I}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{11}^c & a_{21}^c & a_{31}^c & a_{41}^c \\ a_{12}^c & a_{22}^c & a_{32}^c & a_{42}^c \\ a_{13}^c & a_{23}^c & a_{33}^c & a_{43}^c \\ a_{14}^c & a_{24}^c & a_{34}^c & a_{44}^c \end{bmatrix}$$

Properties :-

$$\rightarrow A^{-1}A = I$$

$$\rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

$$\rightarrow (AB)^T = B^T A^T$$

$$\rightarrow |AB| = |A| |B|$$

$$\rightarrow |A^{-1}| = \frac{1}{|A|}$$

$$\rightarrow |A^T| = |A|$$

\rightarrow Diagonal Matrix

→ Diagonal Matrix

All non-zero elements
along diagonal.

→ Symmetric Matrix

$$A = A^T$$

→ Orthogonal Matrix

$$RR^T = I = R^T R$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$
0 ✓	0 ✓	0 ✗
S	S	S
D	D	D

Practice :- ① Compute the determinant of

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

② Inverse of $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

⑤ Eigen Values and Eigen Vectors

Those non-zero vectors whose direction do not change when multiplied by the matrix.

$$Aa = \lambda a$$

Matrix → A, eigen value → λ , eigen vector → a

$$(A - \lambda I)a = 0$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

Q:- Compute the eigen values and eigen vectors of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

⑥ Eigen Value Decomposition

If A is symmetric matrix,

$$A = A^T$$

$$A = Q D Q^T$$

$Q \rightarrow$ Orthogonal Matrix

$D \rightarrow$ Diagonal Matrix

Eigen Vectors \rightarrow Columns of Q .

Eigen Values \rightarrow Diagonal elements of D .

Geometric Transformations :-

① 2D Linear Transformations

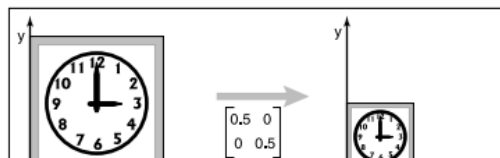
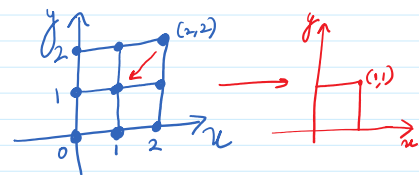
Changing a 2D vector using a 2x2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

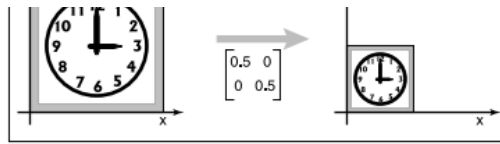
Scaling \rightarrow scale along coordinate axes.

$$\text{Scale } (b_x, b_y) = \begin{bmatrix} b_x & 0 \\ 0 & b_y \end{bmatrix}$$

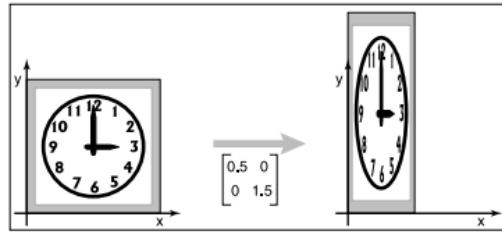
$$\begin{bmatrix} b_x & 0 \\ 0 & b_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_x x \\ b_y y \end{bmatrix}$$



$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

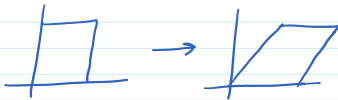


$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Shearing :-

"shear" pushing



shear matrices

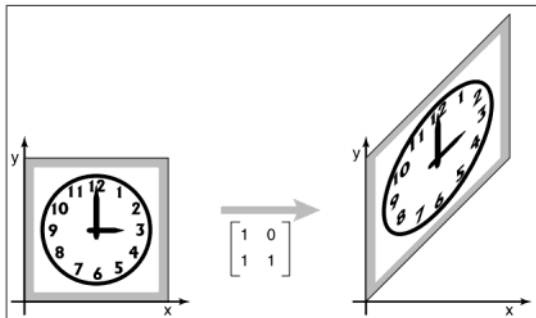
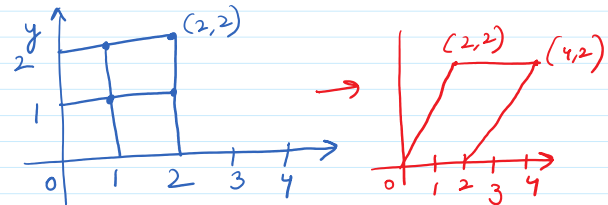
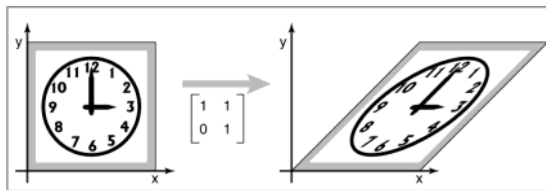
horizontal

vertical

$$\text{shear-}x(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

$$\text{shear-}y(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

$$s = \tan \phi$$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 7 & -1 & -2 \\ -5 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

Given matrix,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A - \lambda I = 0$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - 1 = 0$$

$$2 - 2\lambda - \lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

Solutions are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\boxed{\frac{3 \pm \sqrt{5}}{2} = \begin{bmatrix} 2.618 \\ 0.382 \end{bmatrix}}$$

Exact eigen values

Associated Eigen Vectors :-

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = 2.618 \Rightarrow \begin{bmatrix} 2-2.618 & 1 \\ 1 & 1-2.618 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(x, y) = (0.8507, 0.5257)$$

$$\lambda_2 = 0.382 \Rightarrow \begin{bmatrix} 2-0.382 & 1 \\ 1 & 1-0.382 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(x, y) = (-0.5257, 0.5807)$$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is a symmetric matrix ($A=A^T$) with eigen values (2.618, 0.382) and one possible set of eigen vectors $\begin{bmatrix} 0.8507 \\ 0.5257 \end{bmatrix}$ $\begin{bmatrix} -0.5257 \\ 0.5807 \end{bmatrix}$

So it can be decomposed in the form,

$$A = Q D Q^T$$

where $Q \rightarrow$ orthogonal matrix
 $D \rightarrow$ diagonal matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} 2.618 & 0 \\ 0 & 0.382 \end{bmatrix}}_{\text{diagonal matrix (D)}} \underbrace{\begin{bmatrix} 0.8507 & 0.5257 \\ -0.5257 & 0.8507 \end{bmatrix}}_{Q^T}$$

using eigen vectors
[using eigen values]

"Eigen Value Decomposition"

Singular Value Decomposition:-

Generalization of symmetric eigen value decomposition to non-symmetric matrices.

$$A = USV^T$$

Both are orthogonal but not same.

To solve for SVD,

$$\begin{aligned} M &= AA^T \\ &= (USV^T)(USV^T)^T \\ &= (USV^T)(VS^T U^T) \end{aligned}$$

$$M = US^2U^T$$

"M" is now a symmetric matrix.

So, applying method of eigen value decomposition.

$$\begin{aligned} N &= A^T A \\ &= (USV^T)^T (USV^T) \\ &= (VS^T U^T) (USV^T) \end{aligned}$$

$$\begin{aligned}
 &= (USV^T)' (USV^T) \\
 &= (VS^T U^T) (USV^T) \\
 &N = VS^2V^T
 \end{aligned}$$

Q: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ← Non symmetric matrix

$M = AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ Which was earlier computed.

$\Rightarrow U = \begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix}$

$N = A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ symmetric?

$V = \begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix}}_U \begin{bmatrix} 1.618 & 0 \\ 0 & 0.618 \end{bmatrix} \underbrace{\begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix}}_V$$